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ON THE RESOLVABILITY OF SINUSOIDS OF NEARLY EQUAL FREQUENCY

by

R.V. Webber

DEPARTMENT OF COMMUNICATIONS
MINISTÈRE DES COMMUNICATIONS

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10 R.V. Webber

(Radio and Radar Branch)

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by

R.V. Webber

ABSTRACT

The possibility of resolving two sinusoids has been investigated for the case where $\Delta f \Delta \tau < 1$ where Δf is the difference in the frequencies of the two sinusoids and $\Delta \tau$ is the time interval during which data may be taken. The chances of successful resolution is very dependent on the difference in the phase constant of the two sinusoids. The chances are best when the sinusoids are in phase and are worst when the sinusoids are out of phase. Noise makes resolution more difficult. The method of investigation consisted of drawing contours of the rms residual resulting from the fitting of two sinusoids, by least squares, to the data. With no noise in the data the procedure corresponds to finding the Fourier series which best fits the data. With noise in the data the series so found is an approximation only to the Fourier series for the data.

1. INTRODUCTION

This report deals with the resolution of two sinusoids of nearly equal frequency: that is, the decision as to whether there are two sinusoids in a signal or only one and the determination of the properties of each signal. The bulk of the report is devoted to studying the difficulties with which any decision rule must deal. A possible decision rule is given at the end.

We first discuss how the problem arose and some of the restrictions under which the problem must be solved. We then show that the approach is a special case of a general Fourier analysis. The method of least squares best fit is then described. Following that we describe the surface in frequency space on which a search must be made for the frequencies which give the minimum rms residual. Contours of these surfaces are shown for the fitting of two sinusoids to both a one-sinusoid signal and a two-sinusoid signal. The effect on this surface of noise in the data is shown as well as the effect of varying the signal parameters. The parameters of the derived signals are discussed briefly, and finally, a possible way is described for deciding between a one-sinusoid signal and a two-sinusoid signal.

The problem arises from a consideration of the resolution, in either range or direction, of two closely spaced radar targets when a swept FM sounder is to be used. Consider the range problem first. With one target, a swept frequency CW sounder generates, after demodulation, a sinusoid, $Ae^{i(2ft + \phi)}$, whose frequency, f, is proportional to the range of the target. The amplitude, A, is proportional to the strength of the echo. The phase constant, ϕ, is unknown. With two targets the vector sum of two sinusoids is generated. The difference in the frequencies of the two sinusoids is proportional to the difference in the ranges of the two targets. Thus, the problem of resolving the ranges of two closely spaced targets becomes the problem of resolving the frequencies of two sinusoids whose frequencies are nearly equal. Now consider the problem of resolution in direction of arrival. With a linear array of antennas and a swept frequency CW sounder, a single target generates, after demodulation, a spatial sinusoid $Ae^{i(2\pi f_x x + \phi_0)}$, where x is distance along the array. The spatial frequency, f_x, is given by $f_x = (\cos \theta)/\lambda$ where θ is the angle between the array axis and the target. Two targets generate the vector sum of two spatial sinusoids along the array. The difference in the spatial frequencies of the two sinusoids is proportional to the difference in the cosines of the angles of the two targets. Thus the problem of resolving the directions of two closely spaced targets becomes the problem of resolving two sinusoids whose spatial frequencies are nearly equal. For this application, "time" as used in this paper would be replaced by "distance along the array".

Several workers have considered the general problem of resolving signals, usually in terms of resolving radar targets. Woodward[1] and Seibert[2] discuss it in terms of ambiguity functions. Ksienski and McGhee[3,4] investigated angular resolution beyond the Rayleigh limit. Nilsson[5] examined the optimum range resolution of radar signals. Gething[6] presented a method of resolving the elevation angles of several incoming signals. Other relevant papers are those by Helstrom[7], Buck and Gustincic[8] and Lichtenstein and Young[9].

It is assumed that the data are digitized and complex, that is, each datum consists of an amplitude and a phase (or a real part and an imaginary part), and that the data has been recorded in a limited time interval. We shall try to resolve two sinusoids of small frequency separation. For illustration purposes we shall assume that the signal duration is one second and that the frequencies of the two sinusoids are separated by 0.2 Hertz.

2. SOME COMMENTS ON FOURIER ANALYSIS

Consider first the classical discrete Fourier analysis. The discrete Fourier series which represents the sampled data, $y(t_j)$, is

$$y(t_j) = \sum_{n=0}^{N-1} Z_n e^{i\frac{2\pi nj}{N}} \tag{1}$$

where $t_j = j\Delta t$ is the time, N is the number of terms in the expansion and $Z_n = A_n e^{i\phi_n}$ where A_n and ϕ_n are the amplitude and the phase constant of the n^{th} component. In the usual discrete Fourier analysis, the complex constants, Z_n, are selected to make the series fit the data best in the sense of least squares[10]. The number of terms, N, in the Fourier series is equal to the number of data and Z_n are then computed by the expression

$$Z_n = \frac{1}{N}\sum_{j=0}^{N-1} y(t_j) e^{-i\frac{2\pi nj}{N}} \tag{2}$$

Using this expression each Z_n is found separately and independently of all other Z_n.

The frequency of the n^{th} component is

$$f_n = \frac{n}{L} = \frac{n}{N\Delta t} \tag{3}$$

where L in seconds is the total time interval during which data are taken, and the frequency difference, Δf, of any two successive components is the same as the resolution and is given by the expression

$$\Delta f\ L = 1 \tag{4}$$

Thus, if we have only one second of data the usual discrete Fourier analysis will not resolve components whose frequencies are separated by less than one Hertz.

Now let us turn our attention to a superesolution technique, and show its equivalence to the discrete Fourier analysis. If we have a Fourier series in which N, the number of terms, if greater than the number of data, the formalism can still be preserved by adding zeros to the data to bring the total number up to N, and hence increasing L and the resolution (decreasing Δf in Equation 4). However, there are now too many unknowns and restrictions (in this case the assumption that the amplitude of most of the terms is zero) must be introduced for a solution. Our objective is to be

able to resolve two sinusoids for which the product ΔfL is as low as 0.2 or perhaps even 0.1. The purpose of the present investigation is to learn the conditions under which we can hope to reach this goal.

The approach is to fit sinusoids to the data by the method of least squares. We will consider only the cases where all the energy is confined to a band of about two Hertz. There may be only one sinusoid or there may be two sinusoids. For the present we assume that there is no noise in the data.

The Fourier analysis is now much simplified. There are at most two non-zero A_n.

We now find the two sinusoids

$$S_j = A_j e^{i(2\Pi f_j t + \phi_j)}$$
$$S_k = A_k e^{i(2\Pi f_k t + \phi_k)} \tag{7}$$

which gives the least squares best fit to the data. Note that this can be done without knowing the value of N, and that since there is no noise in the data, the rms residue may be reduced to a value of zero.

If there are two components they must be orthogonal in the interval L. Using Equation (3) we see that their indices n_j, n_k must be such that

$$\frac{n_j}{n_k} = \frac{f_j}{f_k} \tag{8}$$

From Equation (8) we observe that the ratio of the frequencies of any two components of a Fourier series is a rational number.

From Equation (8) and the relationship $L = N\Delta t$ we have

$$N = \frac{n_j}{f_j(\Delta t)} = \frac{n_k}{f_k(\Delta t)} \tag{9}$$

We now know all the constants in Equation (1); that is, we know a Fourier series for the data. This series will be orthogonal in the interval $N\Delta t$, but not, in general, in the interval during which data were taken.

With noise in the data our procedure will not correspond strictly to a Fourier analysis. The noise introduces extra components so that the Fourier series must have more than two terms. Moreover, the terms we find will not be independent of the other terms since they are not fitted to the data throughout the whole interval of orthogonality.

However, with a high signal to noise ratio, the extra terms will be small when compared with the signal sinusoids and the error which is intro-

duced by neglecting the small terms will be insignificant. This error will increase as the signal to noise ratio decreases.

3. LEAST SQUARES METHOD

Let the data, y_j, be sampled and let each datum be a complex number. For illustration purposes we assume that we have fifteen data. Let S_1 and S_2 be the sinusoids, $A_1 e^{i(2\pi f_1 t + \phi_{01})}$ and $A_2 e^{i(2\pi f_2 t + \phi_{02})}$, which we are to fit to the data by the method of least squares. Let

$$Q = \frac{1}{N}\Sigma |y_j - S_1 - S_2|^2 \tag{10}$$

The problem is to find S_1 and S_2 such that Q is a minimum. Note that Q is a function of A_1, A_2, ϕ_{01}, ϕ_{02}, f_1, f_2 and each of these must be treated as an independent variable in finding the minimum. The calculation of the minimum is done in two parts.

First Q is minimized with respect to the set (A_1, A_2, ϕ_{01}, ϕ_{02}) for specified values of f_1 and f_2. This is done algebraically by solving the linear complex matrix equation

$$CZ = Y \tag{11}$$

where

$$Z = (Z_1, Z_2)^T \tag{12}$$

$$C = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} \tag{13}$$

$$Y = (Y_1, Y_2)^T \tag{14}$$

$$C_{jk} = \sum_{m=1}^{N} (X^*_j X_k)^{m-1} \tag{15}$$

$$Y_j = \sum_{M=1}^{N} y_m (X^*_j)^{m-1} \tag{16}$$

$$X_j = e^{i2\pi f_j \Delta t} \tag{17}$$

$$Z_j = A_j e^{i\phi_{0j}} \tag{18}$$

and the T in Equations (12) and (14) means "transpose". These equations are derived in Appendix A.

Let Q' be this minimum in Q with respect to (A_1, A_2, ϕ_{01}, ϕ_{02}) for specified values of f_1 and f_2. It follows that a minimum in $Q' = Q'(f_1, f_2)$, or in $\sqrt{Q'}$, with respect to (f_1, f_2) coincides with a minimum in Q with respect to (A_1, A_2, ϕ_{01}, ϕ_{02}, f_1, f_2).

Thus the second step is to minimize $\sqrt{Q'}$ with respect to (f_1, f_2). To find such a minimum the computer calculates values of $\sqrt{Q'}$ in the (f_1, f_2) plane and plots contours. In this work contour plots of $\sqrt{Q'}$ were investigated near a minimum. The computer program would first search for a given contour along the 45° diagonal. If found, the contour would be traced. If not found on the 45° diagonal, a search for the contour was made along the 135° diagonal. The two diagonals were searched because within the circle in which the contours were traced some contours crossed the 45° diagonal which did not cross the 135° diagonal and some crossed the 135° diagonal which did not cross the 45° diagonal.

To simulate a signal the real and imaginary parts of a one-sinusoid signal or of the vector sum of a two-sinusoid signal were generated in the computer for each datum. The time t = 0 was taken to be the middle of the data interval. To simulate Gaussian noise, independent and random Gaussian numbers were generated such that the expected value was zero and the expected rms value was equal to the square root of the mean noise power, σ_n. Separate random numbers were added to the real and the imaginary parts of each datum.

The S/N ratio was found from the ratio of the amplitude of the largest component, which was always one, to σ_n, i.e.

$$S/N(dB) = -20 \log_{10} \sigma_n \tag{19}$$

4. CONTOURS FOR A ONE-SINUSOID SIGNAL

For these contours the simulated signal contains one sinusoid, S_α, and noise when specified. Two sinusoids S_1 and S_2 are fitted to the data, and the residual errors are studied.

The region where contours are drawn is the semicircle above the 45° diagonal as shown in Figure 1. The radius of the semicircle corresponds to 0.3 Hz. This would correspond in operational terms to knowing that wave energy was confined to within 0.3 Hz of f_α, and looking for two frequencies in this band. No contours are drawn below the diagonal since the diagram is symmetric in f_1 and f_2. The frequency of the signal, f_α, is represented by the two orthogonal dashed lines. This would be the objective of our search in an operational situation.

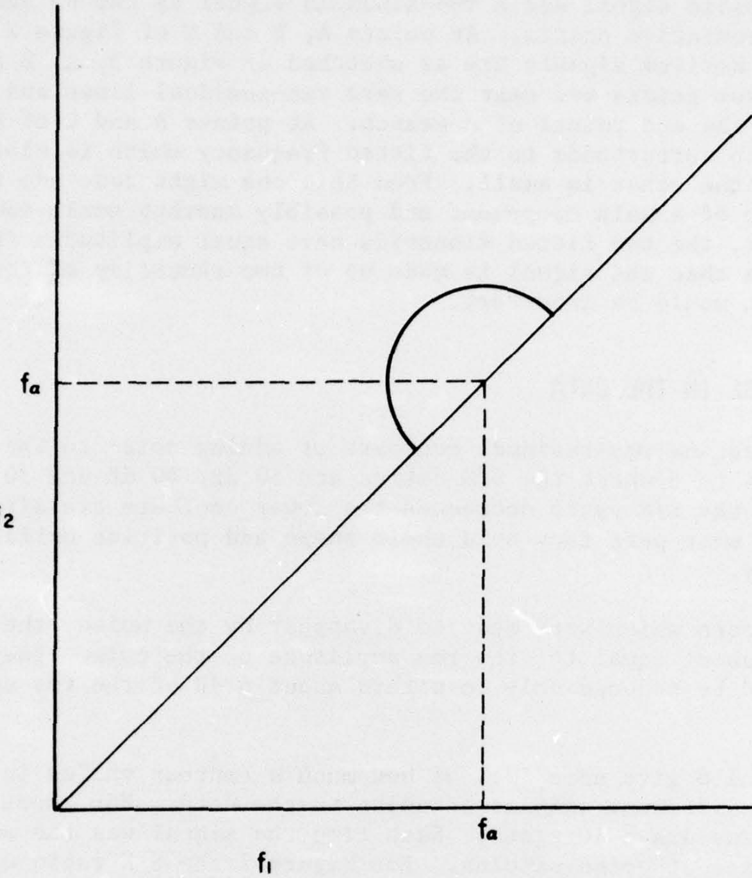


Figure 1. Diagram showing semi-circle in which contours were traced for a one-sinusoid signal

The numbers on the contours in the remaining figures are the height of the contour in dB with respect to the amplitude of the signal. The two orthogonal straight lines are $f_1 = f_\alpha$, $f_2 = f_\alpha$. The contours for which a search was made were, in dB, -54, -46, -40, -30, -26, -20, -14, -10 and -8. For all plots for a one-sinusoid signal the amplitude of the signal was unity and the phase constant was zero. Other values of the phase constant yielded identical plots.

4.1 NOISE FREE DATA

In Figure 2 contours are plotted for a pure one-sinusoid signal which contains no noise. The height, $\sqrt{Q'}$, of the contours with respect to the signal amplitude are given in dB by the expression

$$dB = 20 \log_{10} \sqrt{Q'}$$

If either f_1 or f_2 equals f_α the rms-residual is zero. If the rms-residual is near zero then one, or both, of f_1 or f_2 will be near f_α. However, we

cannot without additional checking use this characteristic to distinguish between a one-sinusoid signal and a two-sinusoid signal as can be seen by considering representative points. At points A, B and C of Figure 2 the amplitudes of the derived signals are as sketched in Figure 3, A, B and C respectively. These points are near the zero rms-residual lines and are representative of the end points of a search. At points A and C of Figure 2 the amplitude which corresponds to the fitted frequency which is closest to f_α is large while the other is small. From this one might conclude that the signal was made up of a main component and possibly another small component. At point B however, the two fitted sinusoids have equal amplitudes from which one might conclude that the signal is made up of two sinusoids of equal amplitudes - which would be incorrect.

4.2 EFFECT ON NOISE IN THE DATA

The effect on the rms-residual contours of adding noise to the data is shown in Figures 4 to 6 where the S/N ratios are 50 dB, 40 dB and 30 dB respectively. As the S/N ratio decreases the lower contours are affected by noise but for the most part they hold their shape and position until they disappear entirely.

Of the contours which were made to disappear by the noise, the highest was equal to, or about equal to, the rms amplitude of the noise itself. The rms-residual could be reduced only to within about 6 dB of the rms amplitude of the noise.

Figures 7 and 8 give some idea of how much a contour varies in shape and position with different samples of noise in the data. For these figures the same contour was drawn 10 times. Each time the signal was the same but with a different set of noise samples. For Figure 7 the S/N ratio was 50 dB. For Figure 8 the S/N ratio was 40 dB. All contours were confined to the shaded areas of the figures.

4.3 EFFECT OF VARYING SIGNAL FREQUENCY

Figure 9, 10 and 11 are for a signal with no noise but with the signal frequency varying linearly in time by 0.01 Hz sec^{-1}, 0.04 Hz sec^{-1} and 0.08 Hz sec^{-1} respectively. The general effect is the same as having noise in the data; the higher variation in frequency corresponding to the lower signal-to-noise ratio.

5. CONTOURS FOR A TWO-SINUSOID SIGNAL

For the two-sinusoid signal the area for which contours were drawn is a segment of a circle above the 45° diagonal centred on the two frequencies, f_α, f_β in the simulated signal. This is illustrated in Figure 12. Presumably, in an operational situation, a semicircle, centred on the centre of the frequency band would be used. The same contours were drawn as for the single-sinusoid signal. Unless otherwise stated $A_\alpha = A_\beta = 1$.

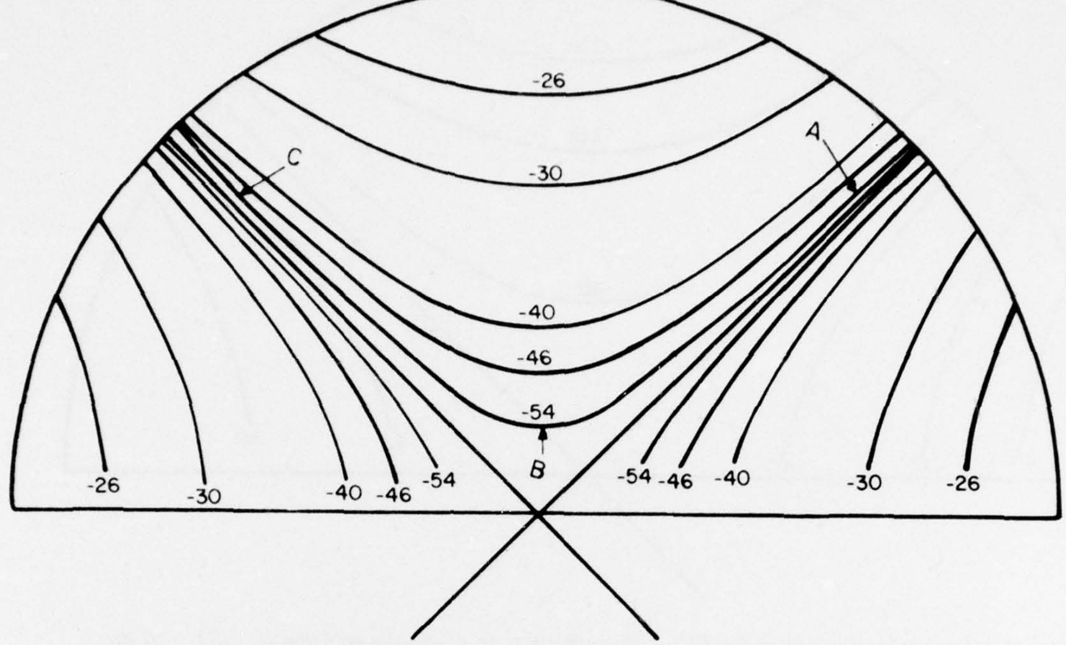


Figure 2. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. No noise in signal.

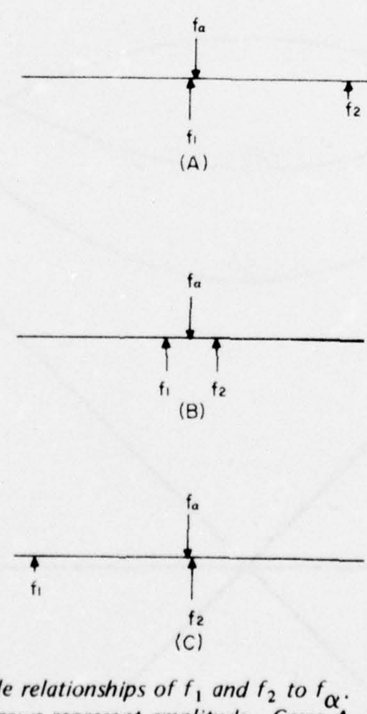


Figure 3. Diagram showing possible relationships of f_1 and f_2 to f_α. The rms-residual is the same in all three cases. The lengths of the arrows represent amplitude. Cases A, B and C correspond to the points A, B and C of Figure 2.

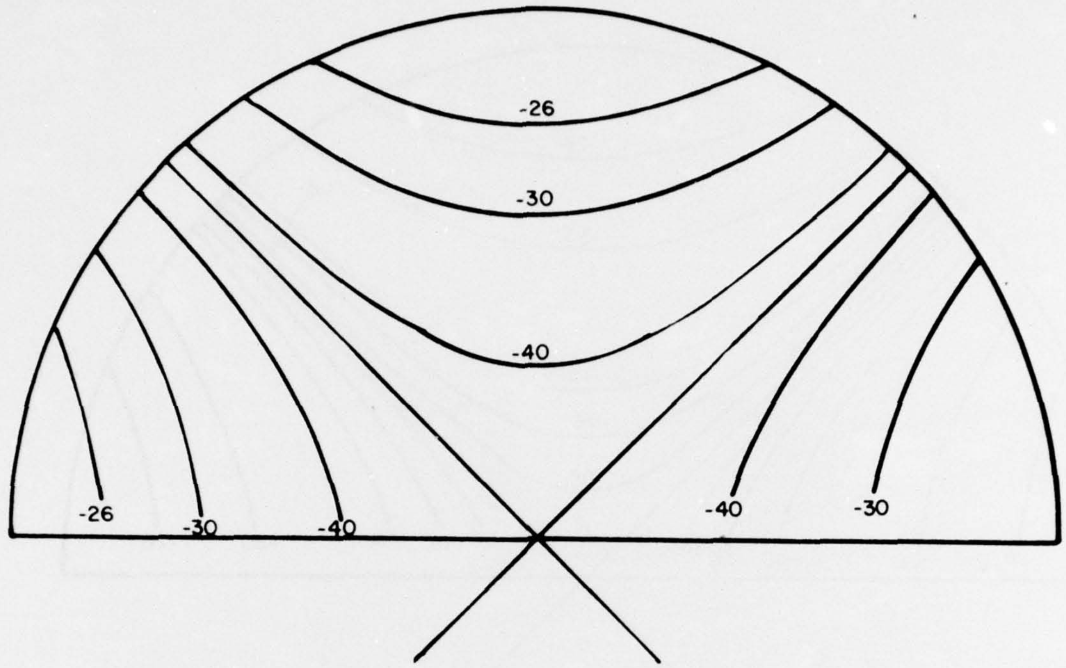


Figure 4. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. S/N = 50 dB.

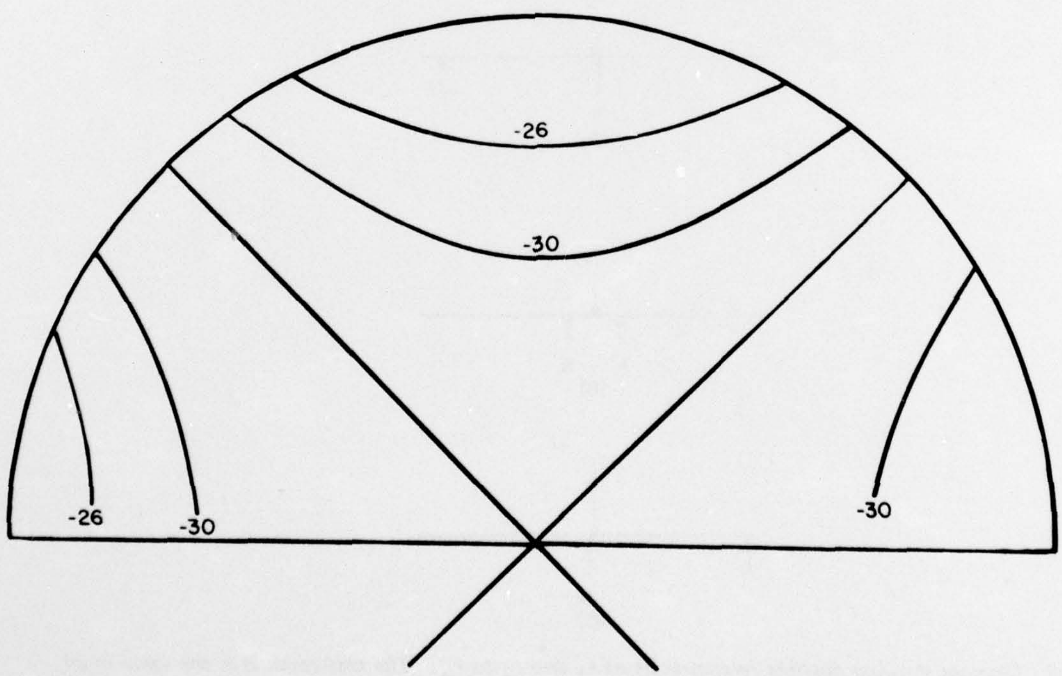


Figure 5. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. S/N = 40 dB.

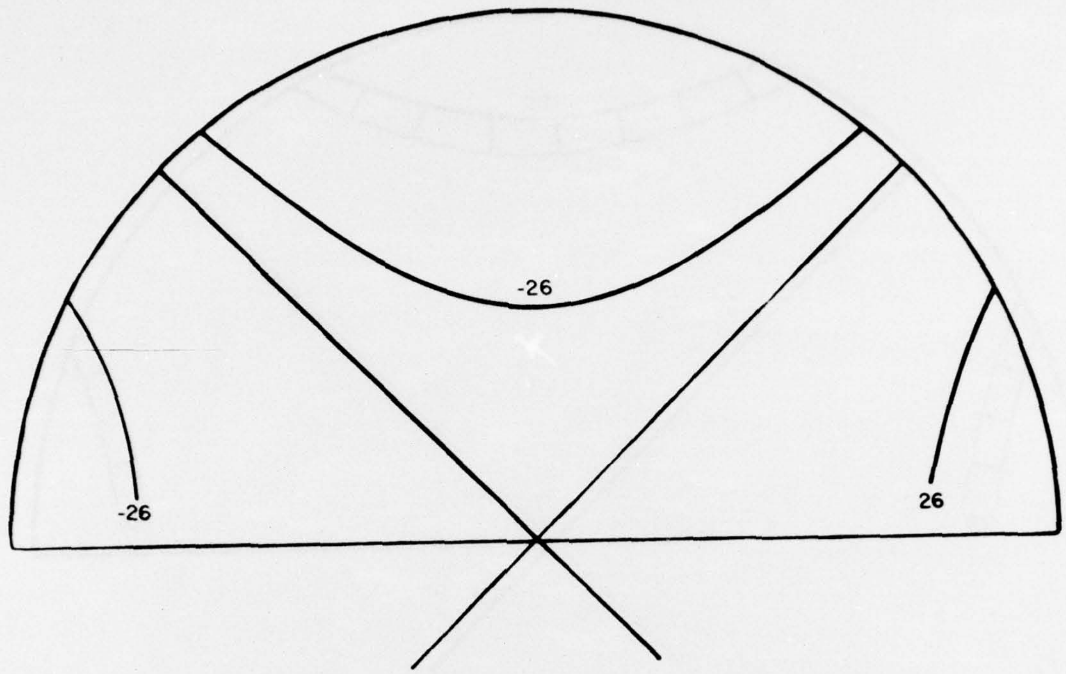


Figure 6. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. SNR = 30 dB.

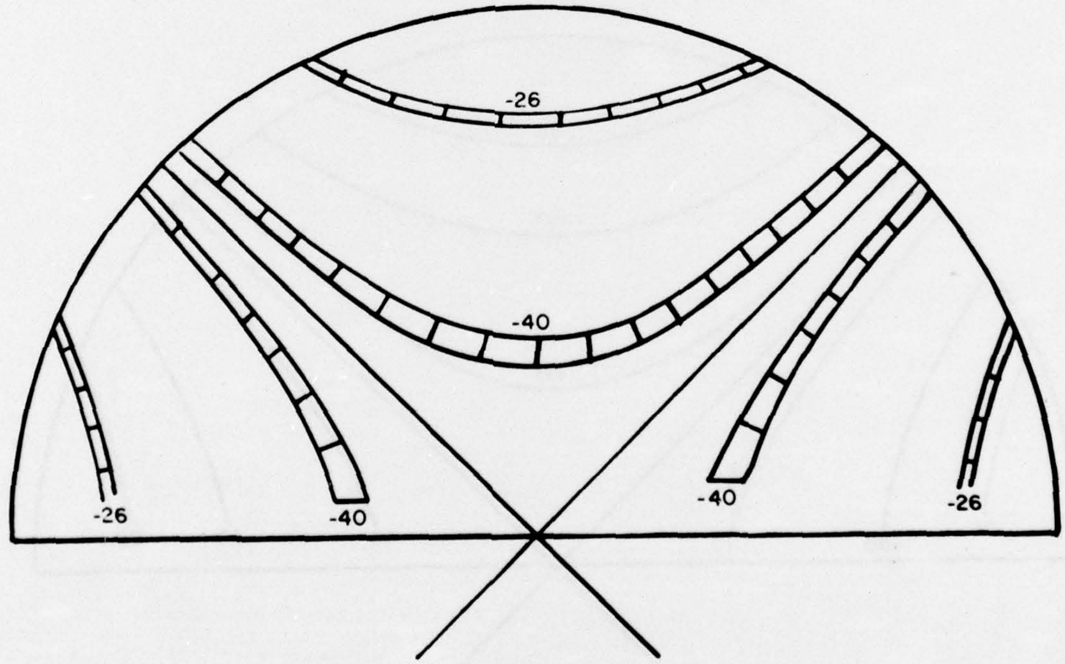


Figure 7. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. Cross-hatched areas show the extent to which contours 0.01 and 0.05 moved around as the noise samples were changed. S/N = 50 dB.

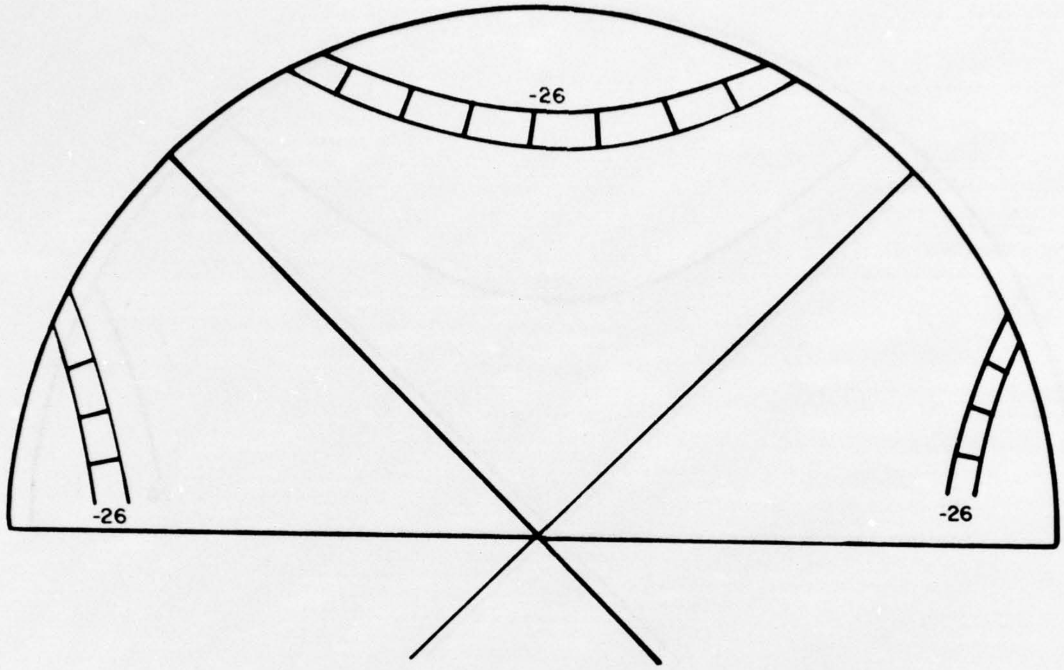


Figure 8. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. Cross-hatched area shows the extent to which contour 0.05 moved around as the noise samples were changed. S/N = 40 dB.

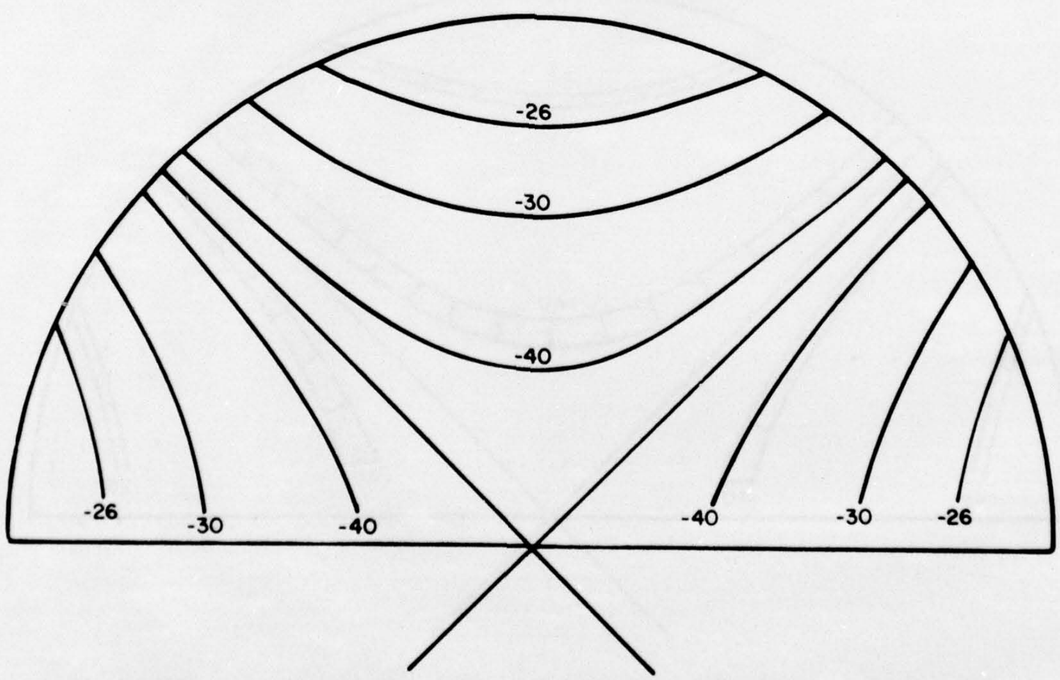


Figure 9. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. No noise. The signal frequency varied by 0.01 Hz sec^{-1}.

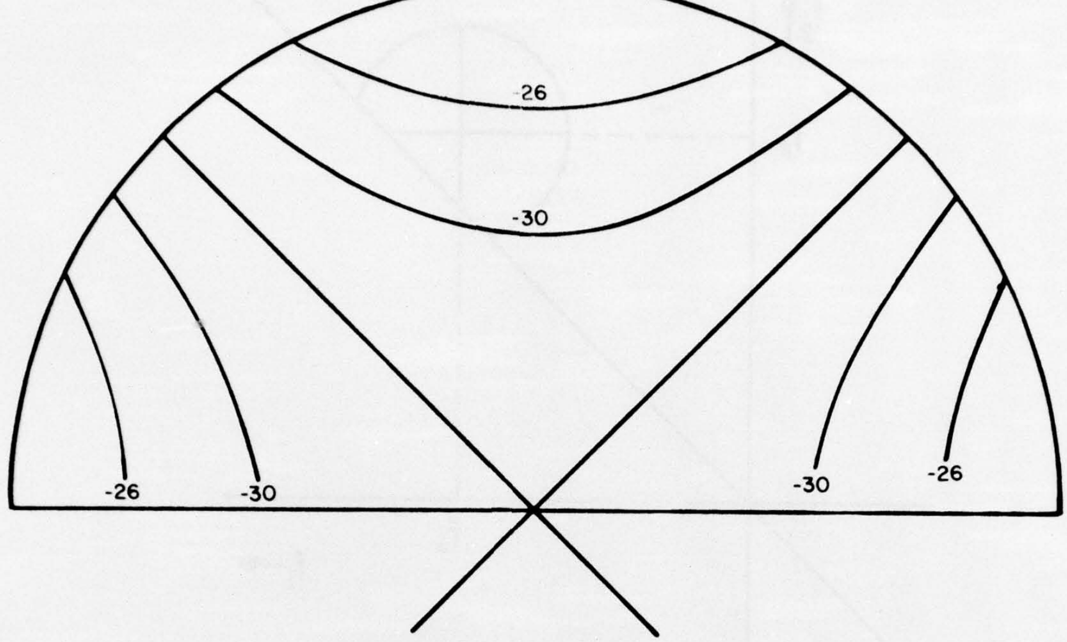


Figure 10. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. No noise. The signal frequency varied by 0.04 Hz sec^{-1}.

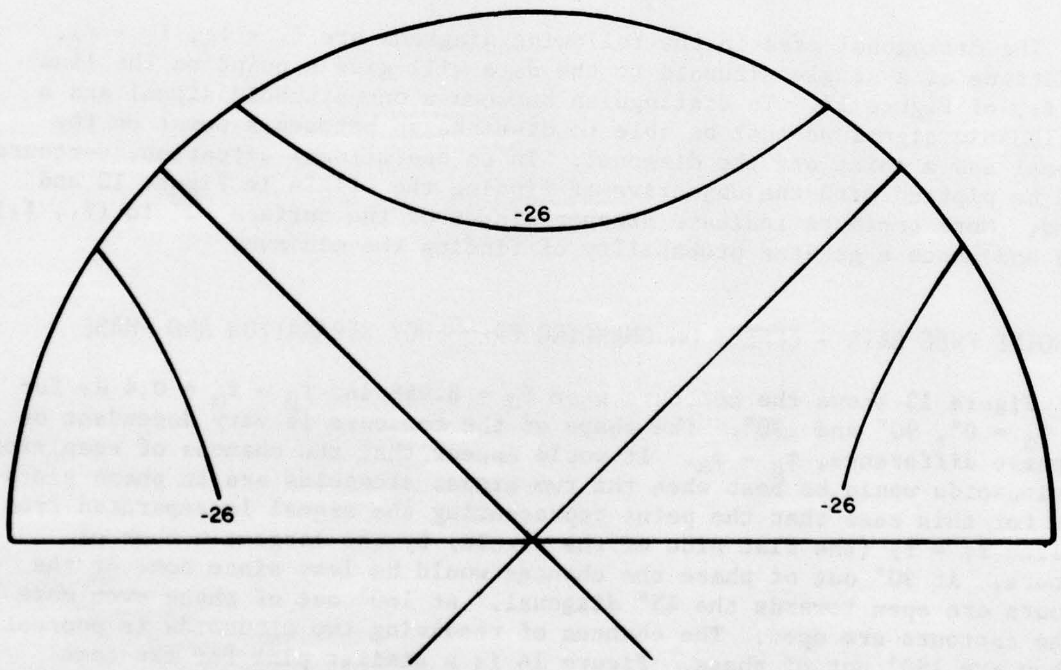


Figure 11. Rms-residual contours for fitting two sinusoids to a one-sinusoid signal. No noise. The signal frequency varied by 0.08 Hz sec^{-1}.

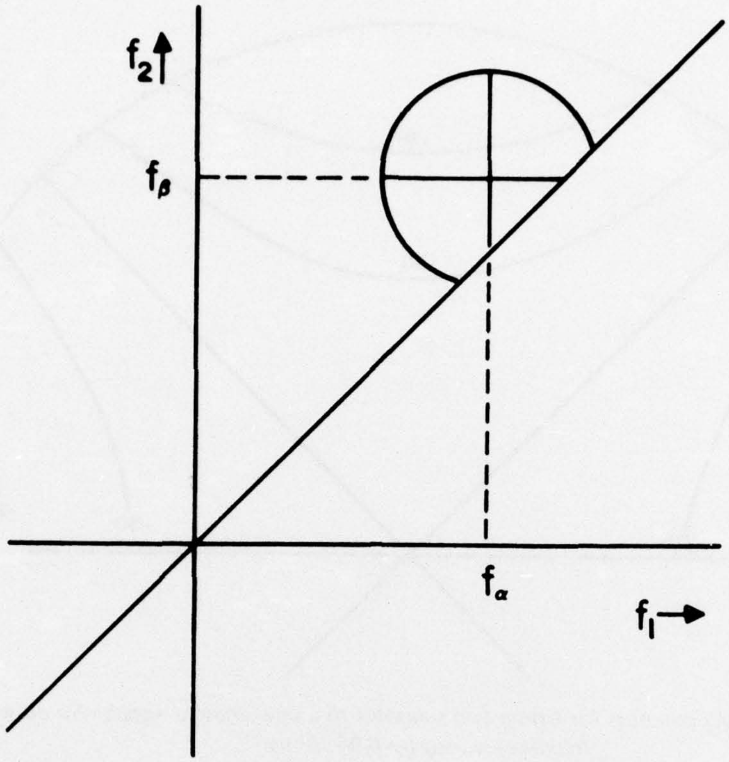


Figure 12. Diagram showing semi-circle in which contours were traced for a two-sinusoid signal.

The orthogonal axes in the following diagrams are $f_1 = f_\alpha$, $f_2 = f_\beta$. The fitting of a single sinusoid to the data will give a point on the line $f_1 = f_2$, of Figure 12. To distinguish between a one-sinusoid signal and a two-sinusoid signal we must be able to distinguish between a point on the diagonal and a point off the diagonal. In an operational situation, contours would be plotted with the objective of finding the origin in Figure 13 and onward. More contours indicate steeper slopes of the surface $\sqrt{Q'}$ in (f_1, f_2) space and hence a greater probability of finding the minimum.

5.1 NOISE FREE DATA - EFFECT ON CHANGING FREQUENCY SEPARATION AND PHASE

Figure 13 shows the contours when $f_\alpha = 8.968$ and $f_\beta - f_\alpha = 0.4$ Hz for $\phi_\beta - \phi_\alpha = 0°$, 90° and 180°. The shape of the contours is very dependent on the phase difference, $\phi_\beta - \phi_\alpha$. It would appear that the chances of resolving two sinusoids would be best when the two signal sinusoids are in phase since it is for this case that the point representing the signal is separated from the line $f_1 = f_2$ (the flat side of the circle) by the largest number of contours. At 90° out of phase the chances would be less since some of the contours are open towards the 45° diagonal. At 180° out of phase even more of the contours are open. The chances of resolving two sinusoids is poorest if they are 180° out of phase. Figure 14 is a similar plot for the same value of f_2 but for $f_\beta - f_\alpha = 0.2$ Hz. The contours are larger and the surface is less steep. The point which represents the signal is less well

separated from the diagonal; this makes resolution more difficult. For $\phi_\beta - \phi_\alpha = 180°$ even the -54 dB contour is open. One would have to go to lower contours to find one that is closed - perhaps to -60 dB or -70 dB.

Figure 15 is again a similar plot for the same value of f_α but for $f_\beta - f_\alpha = 0.1$ Hz. Again the contours are larger and the surface flatter. For $\phi_\beta - \phi_\alpha = 180°$ one would have to go to still lower contours to find one that is closed.

There would be two possible ways to use these contours to aid in finding the approximate position of the center of the axis.

1. To search the surface until the rms residual is below a certain contour level. One would then know that the point found would be inside that contour. Clearly this would only prove the existence of two sinusoids if the point found were on a closed contour, and would probably not give their properties very accurately.

2. Draw rms-residual contours. One could then choose the "center" of the contour. This would give greater accuracy than method (1). Also contours which were not closed could be used. It would, however, be a more lengthy procedure than that of method (1).

Of these two methods, either one could be used if the two signal sinusoids are in phase or even if they are 90° out of phase. With the two signal sinusoids 180° out of phase, however, method (1) would place such demands on the S/N ratio that it would be useless in most cases for $\Delta f \leq 0.2$. There is therefore, little hope of resolving two sinusoids with $\Delta f \Delta t \leq 0.2$ if they are almost 180° out of phase. And this is with no noise in the data.

It was found that the contours depended only on the difference between the signal frequencies and not on the absolute frequencies.

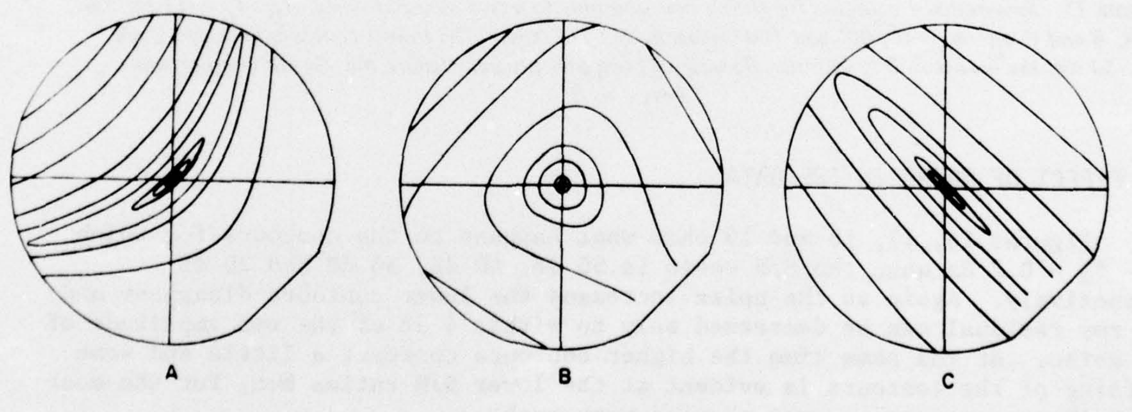


Figure 13. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.4$ Hz; for A, B and C $\varphi_\beta - \varphi_\alpha = 0°$, 90° and 180° respectively. No noise. The lowest contours are at -54 dB.

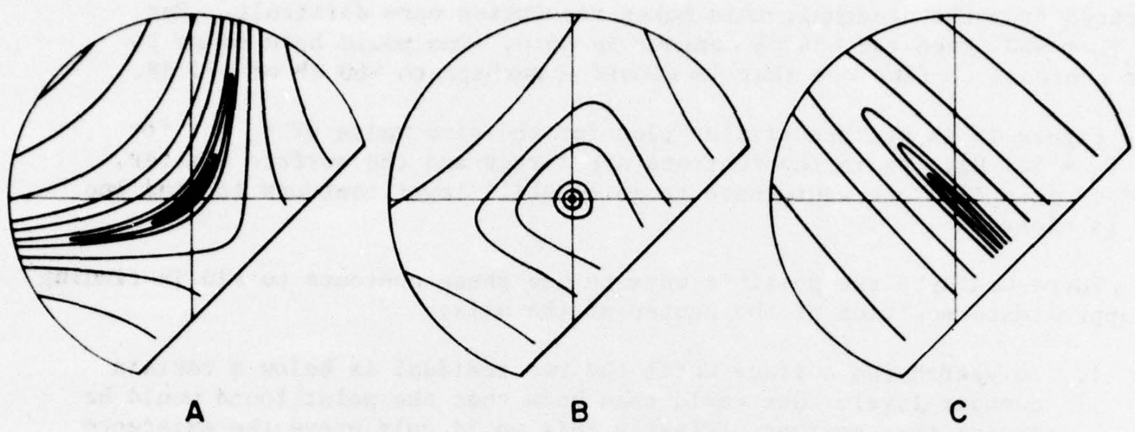


Figure 14. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. No noise. The lowest contours are at -54 dB.

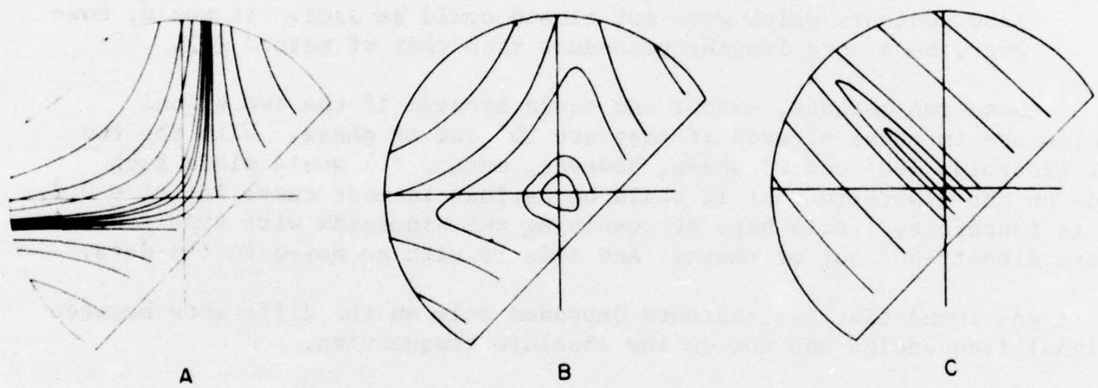


Figure 15. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.1$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. No noise. The lowest contours in A and C are -54 dB; the lowest in B is -46 dB. Because of computer program failure the -54 dB contour was missed in B.

5.2 EFFECT OF NOISE IN THE DATA

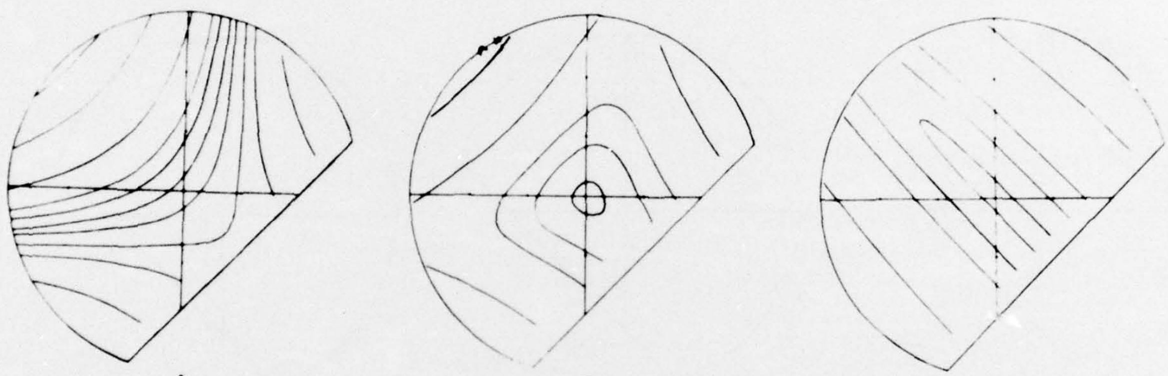
Figures 16, 17, 18 and 19 show what happens to the contours for which $f_\beta - f_\alpha = 0.2$ Hz when the S/N ratio is 50 dB, 40 dB, 30 dB and 20 dB respectively. Again as the noise increases the lower contours disappear and the rms residual can be decreased only to within 6 dB of the rms amplitude of the noise. At the same time the higher contours contract a little and some shifting of the contours is evident at the lower S/N ratios but, for the most part, these contours are not changed very much.

In the figures thus far for two sinusoid signals each diagram corresponds to one set of noise samples, one sample for each datum. To find the variation of the contours as the noise samples change the same contour was traced ten times; each time the S/N ratio was the same but the noise samples

were different. Figures 20 and 21 show the results for S/N ratios of 40 dB and 30 dB, respectively. At a high S/N ratio the location of the contours does not move much; as the S/N ratio decreases the curves move around more. We may note, however, that even at S/N = 30 dB when the two signal sinusoids are in phase or 90° out of phase, the "centers" of all the contours are within 0.1 Hz of the point which corresponds to the signal.

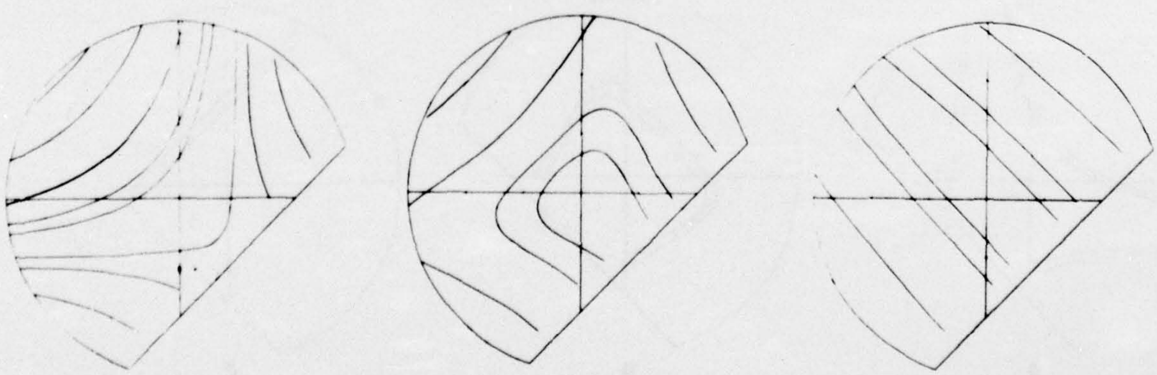
5.3 EFFECT OF VARYING FREQUENCY

Figure 22 shows contours for no noise in the data but for both of the signal frequencies varying by 0.01 Hz sec^{-1}. The lower contours have disappeared. The effect is very similar to adding noise to the data.



A B C

Figure 16. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. S/N = 50 dB. Lowest contours are at -40 dB.



A B C

Figure 17. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. S/N = 40 dB. Lowest contours are at -30 dB.

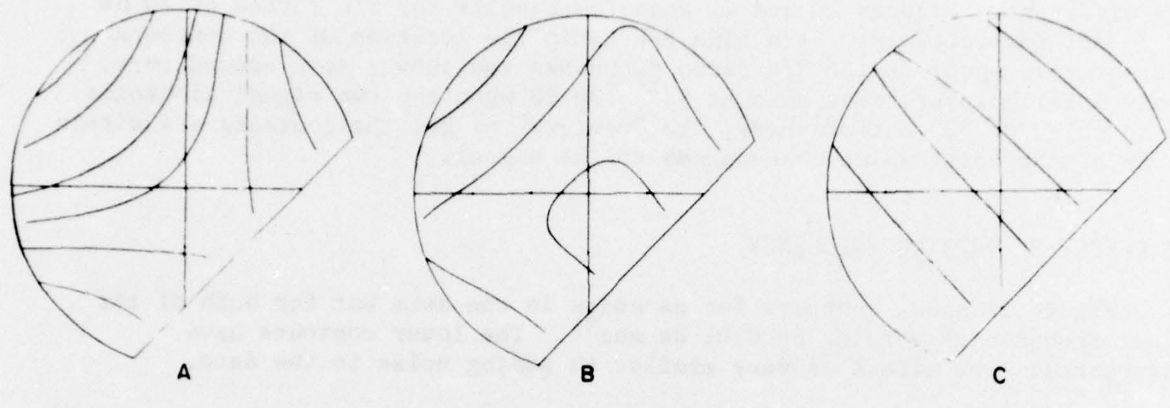


Figure 18. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. S/N = 30 dB. Lowest contours are at -26 dB.

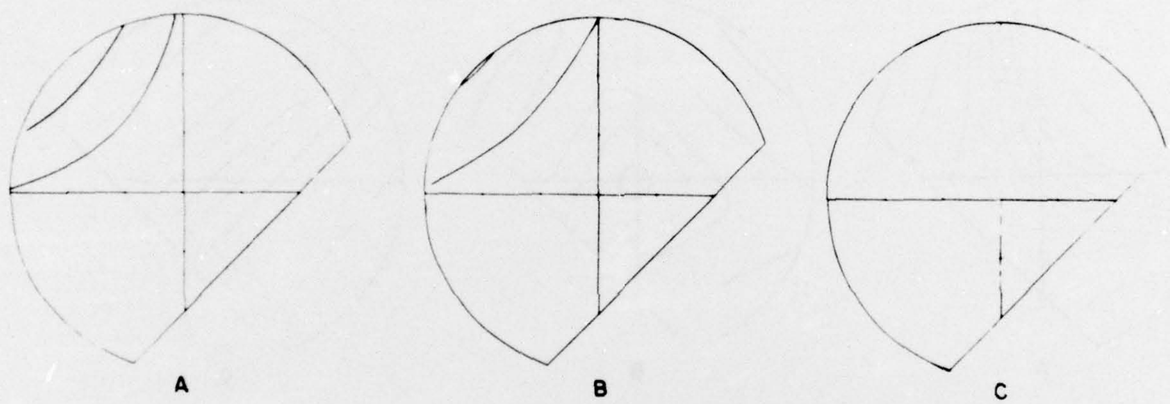


Figure 19. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. S/N = 20 dB. Lowest contours A and B are at -14 dB. No contours found in C.

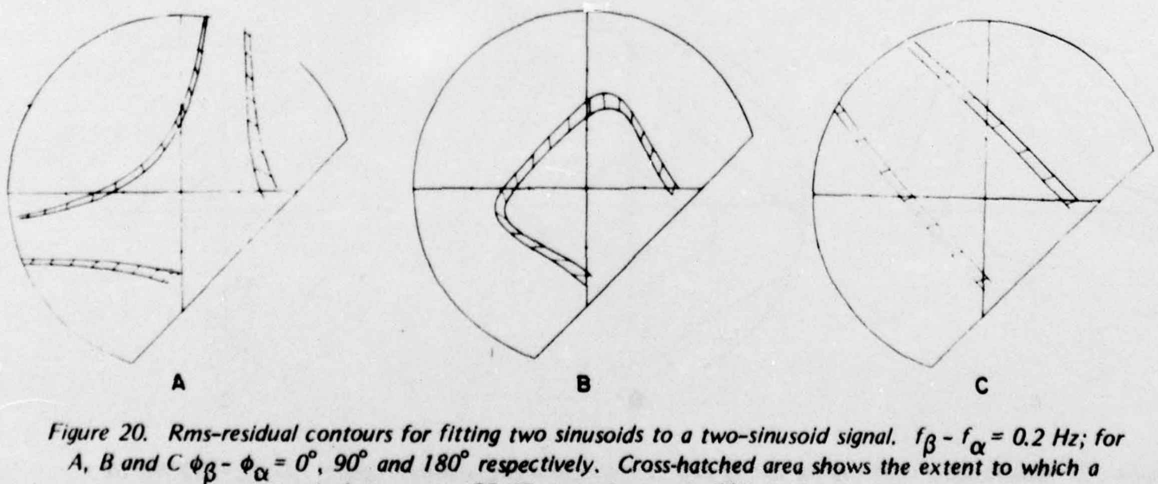
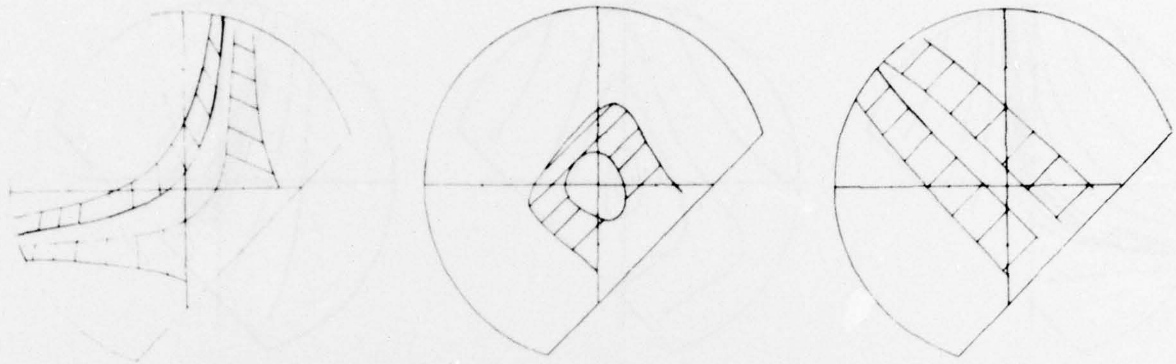
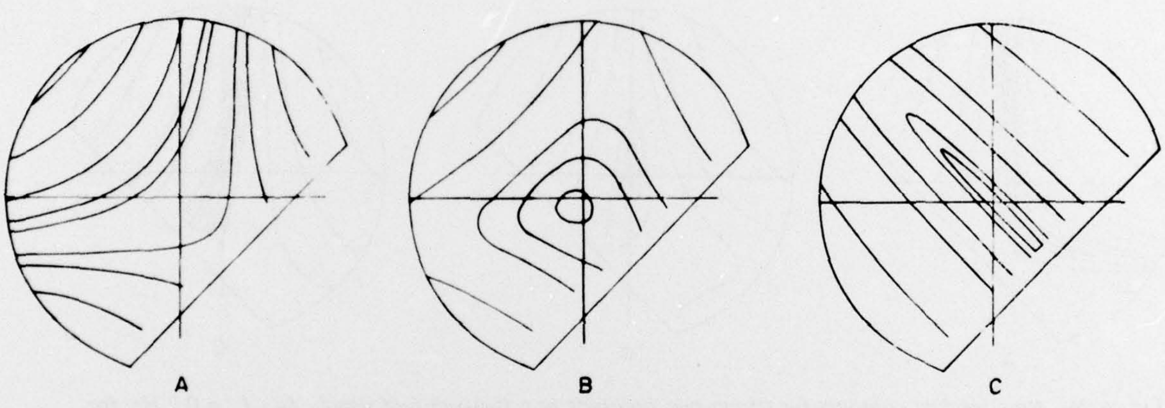


Figure 20. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. Cross-hatched area shows the extent to which a single contour, -26 dB, moved around. S/N = 40 dB.



A B C

Figure 21. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. S/N = 30 dB. The cross-hatched area shows the extent to which a single contour, -26 dB, moved around.



A B C

Figure 22. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \phi_\alpha = 0°$, $90°$ and $180°$ respectively. No noise. Both f_α and f_β vary by 0.01 Hz sec^{-1}. Lowest contour in A is -30 dB; in B -40 dB and in C -46 dB.

5.4 EFFECT OF VARYING AMPLITUDE RATIO

Figures 23 and 24 show contours for no noise in the data and for $A_\alpha = 1.0$ but for $A\beta = 0.5$ and 0.25 respectively. The contours are stretched along the axis which correspond to the signal with the reduced amplitude. The lower the amplitude the greater is the stretching. Stretching corresponds to reducing the slope in the direction of the stretch. This makes resolution more difficult in that direction.

5.5 ANALYSIS OF TWO SUCCESSIVE REALIZATIONS AT THE SAME TIME

Two time-separated sets of measurement of a two sinusoid signal may be analyzed together in such a way that the rms residual of the two realizations

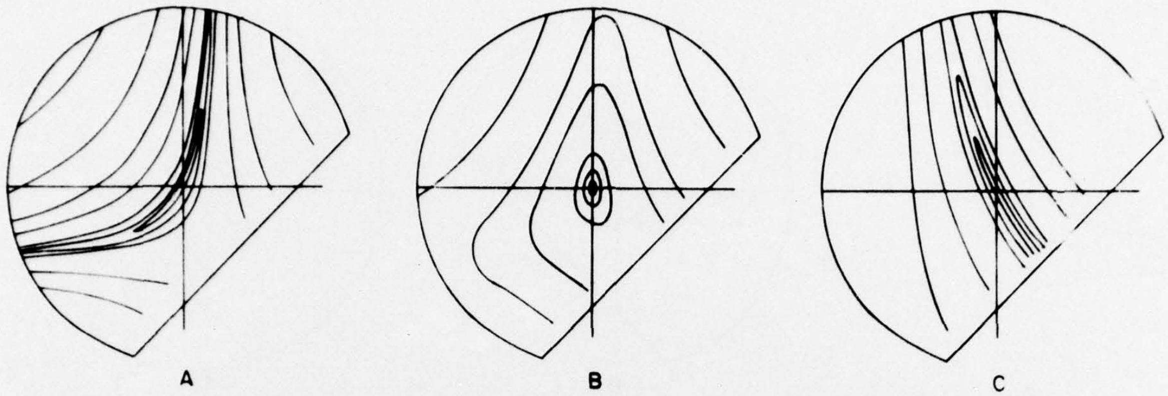


Figure 23. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\phi_\beta - \varphi_\alpha = 0°$, $90°$ and $180°$ respectively. No noise. $A_\alpha = 1.0$; $A_\beta = 0.5$. Lowest contours are at -54 dB.

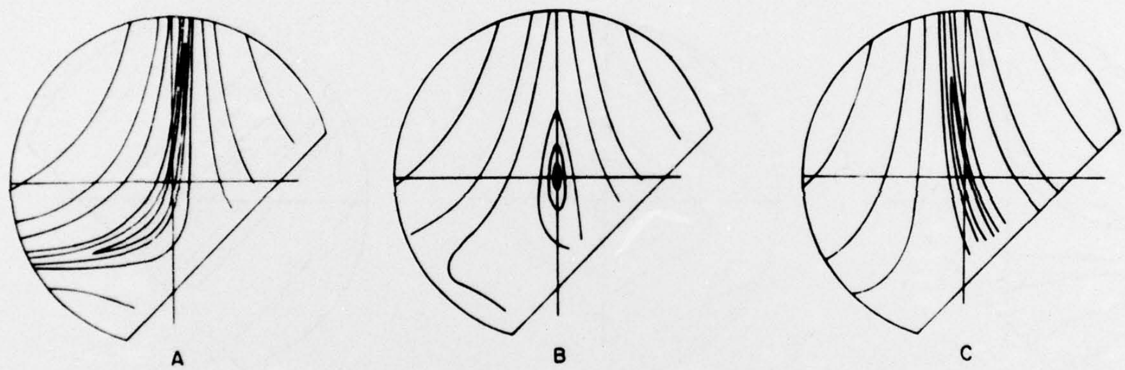


Figure 24. Rms-residual contours for fitting two sinusoids to a two-sinusoid signal. $f_\beta - f_\alpha = 0.2$ Hz; for A, B and C $\varphi_\beta - \varphi_\alpha = 0°$, $90°$ and $180°$ respectively. No noise. $A_\alpha = 1.0$; $A_\beta = 0.25$. Lowest contours are at -54 dB.

is a minimum. The difference between the phase constants of the two sinusoids in the first realization will, in general, be different from that in the second. The result is identical to the result for a single realization whose difference in the phase constant is intermediate between the two which are processed. Since the maximum difference in the phase constants is 180° the difference in the phase constants of the equivalent single realization will, in general, be less than 180°.

For example, two realizations were processed together, one for which $\phi_\beta - \phi_\alpha = 0°$ the other for which $\phi_\beta - \phi_\alpha = 90°$. The contours which were obtained were identical to those which were obtained by processing a single realization for which $\phi_\beta - \phi_\alpha = 60°$.

Equations which relate the equivalent realization to the two realizations which are processed are derived in Appendix B.

By processing two realizations together we could hope to improve the resolvability of two sinusoids for it is much less likely that two realizations will both be near to 180° out of phase than it is that one will be. The equivalent single realization is, therefore, less likely to be near to 180° out of phase than is a single realization.

A disadvantage of processing two realizations together is that two data intervals are required. Thus the time is doubled and, for a given Δf, $\Delta f \Delta \tau$ is doubled where $\Delta \tau$ is the total time during which data are taken. This tends to nullify any advantage that might be gained since one of our main objectives is to keep $\Delta f \Delta \tau$ small.

5.6 THE AMPLITUDES OF THE FITTED SINUSOIDS

The derived amplitudes of the fitted sinusoids vary in a complex manner. They depend on (1) the relation of the frequencies of the fitted sinusoids to the frequencies of the signal sinusoids, (2) the difference between the phase constants of the signal sinusoids, (3) the height of the contour being followed and (4) the amount of noise in the data.

Consider the case for which the amplitudes of the sinusoids in the simulated signal are equal and with no noise. For this case it was found that the amplitudes of the fitted sinusoids are practically always unequal. Sometimes the one and sometimes the other may be larger. Both may be smaller than the amplitudes of the signal sinusoids or both may be larger, by several hundred per cent, or one may be smaller and the other larger. However, they are always about equal on the 135 - 315° diagonal. On the 135° radius this corresponds to the relation between (f_α, f_β, f_1, f_2) as sketched in Figure 25a.

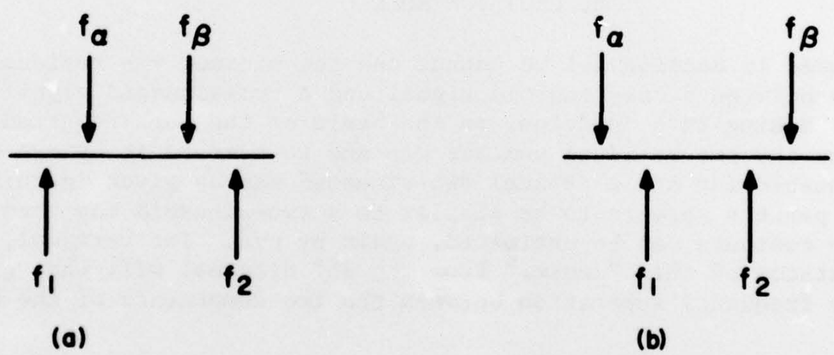


Figure 25.

Where $f_1 - f_\alpha$ is negative and $f_2 - f_\beta$ is positive but both are of equal magnitude. On the 315° radius this corresponds to the relation shown in Figure 25b where $f_1 - f_\alpha$ is positive and $f_2 - f_\beta$ is negative and, again, both are of equal magnitude. Of course, fewer contours exist on the 315° radius than on the 135° radius.

Noise in the data caused variations in the amplitudes of the fitted sinusoids but the general pattern of the results was unchanged.

5.7 THE PHASES OF THE FITTED SINUSOIDS

5.7.1 Noiseless Data

When the two signal sinusoids are in phase the two fitted sinusoids are always in phase with the signal sinusoids for the lower contours which are the important ones since a search presumably ends up on these. For higher contours the two fitted sinusoids may be in phase with the signal sinusoids or one may be in phase and the other out of phase. The shift from in phase to out of phase always occurs off the contour diagram. It is suggested that these higher contours have more than one branch and that for a given branch the fitted sinusoids are always in phase or are always out of phase.

When the signal sinusoids are 90° out of phase, the phase difference between the fitted sinusoid varies from 47° to 170° for the higher contours where, again, there appears to be more than one branch to a contour. For the lower contours their phase difference is closer to 90°.

When the signal sinusoids are 180° out of phase the fitted sinusoids are always in phase with them.

5.7.2 Noisy Data

Putting noise in the data, such that the S/N ratio was as low as 30 dB, had little effect on the phases of the fitted sinusoids when the signal sinusoids were either in phase or 180° out of phase. When the signal sinusoids were 90° out of phase, noise had some effect on the phases of the fitted sinusoids but the general pattern of the results was unchanged.

6. DECISION RULE

As discussed in Section 4.1 we cannot use the minimum rms residual alone to decide between a one-sinusoid signal and a two-sinusoid signal. A possible way of making this decision, on the basis of the work reported here, would be to draw the rms residual contour map and to compare it by eye with a typical one-sinusoid map and a typical two-sinusoid map as given in this paper. If the pattern appears to be similar to a two-sinusoid map then the "center" of the contours can be estimated, again by eye. The vertical, or horizontal, distance of this "center" from the 45° diagonal will then give an estimate of the frequency separation between the two components of the signal.

7. SUMMARY

The possibility of resolving two sinusoids for which $\Delta f \Delta \tau \ll 1$ has been investigated. Contours of constant rms-residual were generated by fitting, using the method of least squares, of two sinusoids to a one-sinusoid signal as well as to a two-sinusoid signal. From these contours it was concluded that the chances of the successful resolution of two sinusoids depends markedly on the difference of their phase constants. It should be easiest if they have the same phase constants and most difficult if their phase differ

by 180°. Noise eliminates the lower contours and hence decreases the precision. The rms-residual can be reduced nearly to the rms noise amplitude but no further. The effect of changing the ratio of the amplitudes of the signal sinusoids and of changing their frequencies was investigated. With no noise in the data our procedure corresponds to the finding of a Fourier series which fits the data. With noise in the data, however, the results correspond only approximately to the Fourier series for the data.

A possible decision rule would be to draw contour maps of the rms-residuals and to compare them with typical maps of a one-sinusoid signal and a two-sinusoid signal.

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APPENDIX A

Derivation of Equations (11) to (18) of the Text

Let
$$S_j(t_m) = A_j e^{i(2\pi f_j t_m + \phi_{oj})} \tag{A1}$$

$$t_m = (m - 1)\Delta t; \; (m = 1, \text{---}N) \tag{A2}$$

$$X_j = e^{i2\pi f_j \Delta t} \tag{A3}$$

$$Z_j = A_j e^{i\phi_{oj}} \tag{A4}$$

where A_j, ϕ_{oj} and f_j are the amplitude, the phase constant and the frequency, respectively, of sinusoid $S_j(t_m)$, t_m is the time of datum $m(m = 1, \text{---} N)$, Δt is the time interval between successive, equally-spaced data and N is the number of data which were taken.

From equations (A1) to (A4) we have

$$S_j(t_m) = Z_j X_j^{m-1} \tag{A5}$$

Now let
$$P = \sum_{m=1}^{N} |Y_m - S_1(t_m) - S_2(t_m)|^2 \tag{A6}$$

where y_m is the datum which was taken at time t_m.

We must find the value of Z_1 and Z_2 which make P a minimum.

Now
$$P = \sum_{m=1}^{N} [Y_m - S_1(t_m) - S_2(t_m)][Y_m^* - S_1^*(t_m) - S_2^*(t_m)] \tag{A7}$$

So
$$P = \sum_{m=1}^{N} [Y_m - Z_1 X_1^{m-1} - Z_2 X_2^{m-1}][Y_m^* - Z_1^*(X_1^*)^{m-1} - Z_2^*(X_2^*)^{m-1}] \tag{A8}$$

Let $$Z_1 = Z_{1,0} + \epsilon_1 \xi_1 \tag{A9}$$

where $Z_{1,0}$ is the value of Z_1 which makes P a minimum, ϵ_1 is a small, real number and ξ_1 is an arbitrary complex number which is not a function of ϵ_1. Thus at the minimum value of P we have

$$\left(\frac{\partial P}{\partial \epsilon_1}\right)_{\epsilon_1 = 0} = 0 \tag{A10}$$

Now

$$P = \sum_{m=1}^{N} \{[Y_m - Z_{1,0} X_1{}^{m-1} - \epsilon_1 \xi_1 X_1{}^{m-1} - Z_2 X_2{}^{m-1}]$$

$$\times [Y_m^* - Z_{1,0}^* (X_1^*)^{m-1} - \epsilon_1 \xi_1^* (X_1^*)^{m-1} - Z_2 (X_2^*)^{m-1}]\} \tag{A11}$$

So

$$\frac{\partial P}{\partial \epsilon_1} = \sum_{m=1}^{N} \{\xi_1 X_1^{m-1} [Y_m^* - Z_{1,0}^* (X_1^*)^{m-1} - \epsilon_1 \xi_1^* (X_1^*)^{m-1} - Z_2^* (X_2^*)^{m-1}]$$

$$+ \xi_1^* (X_1^*)^{m-1} [Y_m - Z_{10} X_1{}^{m-1} - \epsilon_1 \xi_1 X_1{}^{m-1} - Z_2 X_2{}^{m-1}]\} \tag{A12}$$

$$= 2R\{\xi_1^* \sum_{m=1}^{N} (X_1^*)^{m-1} [Y_m - Z_{1,0} X_1{}^{m-1} - \epsilon_1 \xi_1 X_1{}^{m-1} - Z_2 X_2{}^{m-1}]\} \tag{A13}$$

and

$$\left(\frac{\partial P}{\partial \epsilon_1}\right)_{\epsilon_1 = 0} = 2R\{\xi_1 \sum_{m=1}^{N} (X_1^*)^{m-1} [Y_m - Z_{1,0} X_1{}^{m-1} - Z_2 X_2{}^{m-1}]\} \tag{A14}$$

From equations (A10) and (A14) we have

$$2R\{\xi_1^* \sum_{m=1}^{N} (X_1^*)^{m-1} [Y_m - Z_{1,0} X_1{}^{m-1} - Z_2 X_2{}^{m-1}]\} = 0 \tag{A15}$$

This latter equation is true for any value of ξ_1. The only way that this can be so is for the following equation to be true.

$$\sum_{m=1}^{N} \{(X^*_1)^{m-1} Y_m - Z\, X^{m-1} - Z\, X^{m-1}\} = 0 \tag{A16}$$

or

$$Z_{1,0} \sum_{m=1}^{N} (X^*_1 X_1)^{m-1} + Z_2 \sum_{m=1}^{N} (X^*_1 X_2)^{m-1} = \sum_{m=1}^{N} Y_m (X^*_1)^{m-1} \tag{A17}$$

Equation (A17) is true for any value of Z_2.

Now let

$$Z_2 = Z_{2,0} + \epsilon_2 \xi_2 \tag{A18}$$

and carry through the same procedure again. Equations which are similar to equations (A9) to (A17) will be obtained. The one corresponding to equation (A17) will be

$$Z_1 \sum_{m=1}^{N} (X^*_2 X_1)^{m-1} + Z_{2,0} \sum_{m=1}^{N} (X^*_2 X_2)^{m-1} = \sum Y_m (X^*_2)^{m-1}. \tag{A19}$$

Equation (A19) is true for any value of Z_1.

Now choose $Z_2 = Z_{2,0}$ in equation (A17) and choose $Z_1 = Z_{1,0}$ in equation (A18), and we have the two simultaneous equations

$$\left.\begin{aligned} Z_{1,0} C_{11} + Z_{2,0} C_{12} &= Y_1 \\ Z_{1,0} C_{21} + Z_{2,0} C_{22} &= Y_2 \end{aligned}\right\} \tag{A20}$$

where

$$C_{jk} = \sum_{m=1}^{N} (X^*_j X_k)^{m-1} \tag{A21}$$

$$Y_j = \sum_{m=1}^{N} Y_m (X^*_j)^{m-1} \tag{A22}$$

Solving equations (A20) for $Z_{1,0}$ and $Z_{2,0}$ will then give us the values of Z_1 and Z_2 which makes P a minimum.

Equations (A20), (A21) and (A22) are the same as equations (11) to (18) of the text except that in the text they have been written in matrix form and, for convenience, the zeros have been dropped from the subscripts of $Z_{j,0}$.

APPENDIX B

Derivation of Equations for Analyzing Two Realizations Together

Let an odd number of data be taken at equal intervals of time, Δt. Let the middle datum be at $t = 0$ so that

$$t = m\Delta t \tag{B1}$$

$$m = -M,\ -(M+1),\ \ldots\ -1,\ 0,\ 1,\ \ldots\ M$$

Let the signal be made up of the two sinusoids

$$S_{\alpha,m} = A_\alpha e^{i(2\pi f_\alpha m\Delta t + \phi_\alpha)} \tag{B2}$$

$$S_{\beta,m} = A_\beta e^{i(2\pi f_\beta m\Delta t + \phi_\beta)} \tag{B3}$$

Let an assumed test signal be made up of the two sinusoids

$$S_{1,m} = A_1 e^{i(2\pi f_1 m\Delta t + \phi_1)} \tag{B4}$$

$$S_{2,m} = A_2 e^{i(2\pi f_2 m\Delta t + \phi_2)} \tag{B5}$$

Let $$S_m = (S_{\alpha,m} + S_{\beta,m} - S_{1,m} - S_{2,m}) \tag{B6}$$

then
$$\begin{aligned} |S_m|^2 = {} & |S_{\alpha,m}|^2 + |S_{\beta,m}|^2 + 2R\{S_{\alpha,m}S^*_{\beta,m}\} \\ & + |S_{1,m}|^2 + |S_{2,m}|^2 + 2R|S_{1,m}S^*_{2,m}| \\ & - 2R\{S_{\alpha,m}S^*_{1,m} + S_{\alpha,m}S^*_{2,m} + S_{\beta,m}S^*_{1,m} + S_{\beta,m}S^*_{2,m}\} \end{aligned}$$

where the superscript * signifies the complex conjugate and $R(x)$ means "real part of x".

Let $$B_j = A_j e^{i\phi_\alpha} \tag{B7}$$

$$G_{jkm} = e^{i2\pi(f_j - f_k)m\Delta t} \tag{B8}$$

then

$$|S_m|^2 = A_\alpha^2 + A_\beta^2 + A_1^2 + A_2^2 + 2R\{B_\alpha B_\beta^* G_{\alpha\beta m} + B_1 B_2^* G_{12m}$$
$$- B_\alpha B_1^* G_{\alpha 1m} - B_\alpha B_2^* G_{\alpha 2m} - B_\beta B_1^* G_{\beta 1m} - B_\beta B_2^* G_{\beta 2m}\} \tag{B9}$$

Now

$$\sum_{m=-M}^{M} e^{i\theta m} = 1 + 2\sum_{m=1}^{M} \cos m\theta \tag{B10}$$

Let
$$G_{jk} = \sum_{m=-M}^{M} G_{jkm}$$
$$= 1 + 2\sum_{m=1}^{M} \cos 2\pi(f_j - f_k)\Delta t \tag{B11}$$

We note that G_{jk} is a real number. We may now write

$$\sum_{m=-M}^{M} |S_m|^2 = (2M+1)\ (A_\alpha^2 + A_\beta^2 + A_1^2 + A_2^2) + 2\ G_{\alpha\beta}R(B_\alpha B_\beta^*) + G_{12}R(B_1 B_2^*)$$
$$- G_{\alpha_1}R(B_\alpha B_1^*) - G_{\alpha_2}R(B_\alpha B_2^*) - G_{\beta_1}R(B_\beta B_1^*) - G_{\beta_2}R(B_\beta B_2^*)$$

or

$$\sum_{n=-M}^{M} |S_m|^2 = (2M+1)\ (A_\alpha^2 + A_\beta^2 + A_1^2 + A_2^2) + 2\{G_{\alpha\beta}A_\alpha A_\beta \cos(\phi_\alpha - \phi_\beta)$$
$$+ G_{12}A_1A_2\cos(\phi_1 - \phi_2)$$
$$- G_{\alpha_1}A_\alpha A_1 \cos(\phi_\alpha - \phi_1) - G_{\alpha_2}A_\alpha A_2\cos(\phi_\alpha - \phi_2) - G_{\beta_1}A_\beta A_1\cos(\phi_\beta - \phi_1)$$
$$- G_{\beta_2}A_\beta A_2\cos(\phi_\beta - \phi_2)\} \tag{B12}$$

Now we wish to find the realization, r, which is equivalent to the mean of the two realizations p and q, that is

$$2\left(\sum_{m=-M}^{M}|S_m|^2\right)_r = \left(\sum_{m=-M}^{M}|S_m|^2\right)_p + \left(\sum_{m=-M}^{M}|S_m|^2\right)_q \tag{B13}$$

From equations (B12) and (B13) we get

$$\begin{aligned}
&\{(A_{\alpha p}^2 + A_{\alpha q}^2 - 2A_{\alpha r}^2) + (A_{\beta p}^2 + A_{\beta q}^2 - 2A_{\beta r}^2) \\
&\quad + (A_{1p}^2 + A_{1q}^2 - 2A_{1r}^2) + (A_{2p}^2 + A_{2q}^2 - 2A_{2r}^2)\}(2M + 1)\} \\
&+ 2G_{\alpha\beta}\{A_{\alpha p}A_{\beta p}\cos(\phi_{\alpha p} - \phi_{\beta p}) + A_{\alpha q}A_{\beta q}\cos(\phi_{\alpha q} - \phi_{\beta q}) - 2A_{\alpha r}A_{\beta r}\cos(\phi_{\alpha r} - \phi_{\beta r})\} \\
&+ 2G_{12}\{A_{1p}A_{2p}\cos(\phi_{1p} - \phi_{2p}) + A_{1q}A_{2q}\cos(\phi_{1q} - \phi_{2q}) - 2A_{1r}A_{2r}\cos(\phi_{1r} - \phi_{2r})\} \\
&- 2G_{\alpha_1}\{A_{\alpha p}A_{1p}\cos(\phi_{\alpha p} - \phi_{1p}) + A_{\alpha q}A_{1q}\cos(\phi_{\alpha q} - \phi_{1q}) - 2A_{\alpha r}A_{1r}\cos(\phi_{\alpha r} - \phi_{1r})\} \\
&- 2G_{\alpha_2}\{A_{\alpha p}A_{2p}\cos(\phi_{\alpha p} - \phi_{2p}) + A_{\alpha q}A_{2q}\cos(\phi_{\alpha q} - \phi_{2q}) - 2A_{\alpha r}A_{2r}\cos(\phi_{\alpha r} - \phi_{2r})\} \\
&- 2G_{\beta_1}\{A_{\beta p}A_{1p}\cos(\phi_{\beta p} - \phi_{1p}) + A_{\beta q}A_{1q}\cos(\phi_{\beta q} - \phi_{1q}) - 2A_{\beta r}A_{1r}\cos(\phi_{\beta r} - \phi_{1r})\} \\
&- 2G_{\beta_2}\{A_{\beta p}A_{2p}\cos(\phi_{\beta p} - \phi_{2p}) + A_{\beta q}A_{2q}\cos(\phi_{\beta q} - \phi_{2q}) - 2A_{\beta r}A_{2r}\cos(\phi_{\beta r} - \phi_{2r})\} \\
&= 0
\end{aligned} \tag{B14}$$

The G's are functions of frequency only. Since equation (B14) must hold for all values of f_{α_1}, f_{β_1}, f_1 and f_2 each expression in braces, { }, must be zero and we have

$$2(A_{\alpha r}^2 + A_{\beta r}^2 + A_{1r}^2 + A_{2r}^2) = A_{\alpha p}^2 + A_{\beta p}^2 + A_{\alpha q}^2 + A_{\beta q}^2 + A_{1p}^2 + A_{2p}^2 + A_{1q}^2 + A_{2q}^2 \tag{B15}$$

$$2A_{\alpha r}A_{\beta r}\cos(\phi_{\alpha r} - \phi_{\beta r}) = A_{\alpha p}A_{\beta p}\cos(\phi_{\alpha p} - \phi_{\beta p}) + A_{\alpha q}A_{\beta q}\cos(\phi_{\alpha q} - \phi_{\beta q}) \tag{B16}$$

$$2A_{1r}A_{2r}\cos(\phi_{1r} - \phi_{2r}) = A_{1p}A_{2p}\cos(\phi_{1p} - \phi_{2p}) + A_{1q}A_{2q}\cos(\phi_{1q} - \phi_{2q}) \tag{B17}$$

$$2A_{\alpha r}A_{1r}\cos(\phi_{\alpha r} - \phi_{1r}) = A_{\alpha p}A_{1p}\cos(\phi_{\alpha p} - \phi_{1p}) + A_{\alpha q}A_{1q}\cos(\phi_{\alpha q} - \phi_{1q}) \tag{B18}$$

$$2A_{\alpha r}A_{2r}\cos(\phi_{\alpha r} - \phi_{2r}) = A_{\alpha p}A_{2p}\cos(\phi_{\alpha p} - \phi_{2p}) + A_{\alpha q}A_{2q}\cos(\phi_{\alpha q} - \phi_{2q}) \quad \text{(B19)}$$

$$2A_{\beta r}A_{1r}\cos(\phi_{\beta r} - \phi_{1r}) = A_{\beta p}A_{1p}\cos(\phi_{\beta p} - \phi_{1p}) + A_{\beta q}A_{1q}\cos(\phi_{\beta q} - \phi_{1q}) \quad \text{(B20)}$$

$$2A_{\beta r}A_{2r}\cos(\phi_{\beta r} - \phi_{2r}) = A_{\beta p}A_{2p}\cos(\phi_{\beta p} - \phi_{2p}) + A_{\beta q}A_{2q}\cos(\phi_{\beta q} - \phi_{2q}) \quad \text{(B21)}$$

Now in the example mentioned in section V5 we assigned the values.

$$A_{\alpha p} = A_{\beta p} = A_{\alpha q} = A_{\beta q} = A_{\alpha r} = A_{\beta r} = 1 \quad \text{(B22)}$$

In which case, by equation (B16), the cosine of the phase difference of the equivalent realization is equal to the mean of the cosines of the phase differences of the two realizations which are processed. This checks with the example for in that example

$$\begin{aligned} \phi_{\beta p} - \phi_{\alpha p} &= 0° \\ \phi_{\beta q} - \phi_{\alpha q} &= 90° \end{aligned} \quad \text{(B23)}$$

then from equation (B16) we have

$$\cos(\phi_{\alpha r} - \phi_{\beta r}) = \tfrac{1}{2} \quad \text{(B24)}$$

so

$$(\phi_{\beta r} - \phi_{\alpha r}) = 60°$$